Basic Results on "Power Sets"

If S is a set, the power set P(S) of S is the set of all subsets of S (also sometimes denoted by 2^S). Then P(S) is identified with the set of all functions S$\mapsto$\{0,1\} by subset of S $\Leftrightarrow \chi_S$, where χ_S is the "characteristic function of S" defined by

$$\chi_S(x) = 1 \; if \; x \in S$$
$$\chi_S(x) = 0 \; if \; x \notin S$$

Note that is S is finite, the power set of S contains 2^N elements, where N is the number of elements of S. Hence the same "power set". Of course, if N is finite & N$\geq$0 , then $2^N > N$. In a sense, this inequality remains true for infinite sets:

Theorem: If S is a set, then there is no function F: S$\mapsto$P(S) that is onto.
[So intuitively, P(S) is "larger" than S].

Proof: Given a function F:S$\mapsto$P(S) , define a subset A of S as follows:

$$\mathrm{A} = \{\mathrm{x} \in \mathrm{S} : \mathrm{x} \notin \mathrm{F(x)}\}$$

Note that F(x) is a subset of S so the definition of A makes sense. We will show that A$\notin$F(S). Suppose A=F(y). Then if y$\in$A, by the definition of A, y$\notin$F(y)=A. If y$\notin$ A, then it must be false that y$\notin$F(y) so y$\in$F(y)=A. Either way, a contradiction is obtained . So, A$\neq$F(y) for any y, so F is not onto.

Corollary 1: The power set of $\mathbb{Z}$ is not countable

Corollary 2: The set consisting of everything does not exist.
Reason: If everything formed a set, then it would contain the set of all its own subsets, i.e., it would contain all its own subsets. So there would be a function from the set onto its power set: Take each subset (as an element) to itself and everything else to the empty set. This function is onto the power set (of the everything set). But that is impossible, by the theorem.